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ON THE BALLISTICS ACCURACY AS CHARACTERIZED BY MAXIMAL TRAJECTORY INFORMATION

By

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SEPTEMBER 1970

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ROCK ISLAND, ILLINOIS

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ABSTRACT

The purpose of this study is to introduce a measure by which the information content, as possessed by the trajectory generated by a projectile, pertinent to the capability of a weapon system can be characterized.

With the use of this concept and some appropriate mathematical tools a few immediate results are obtained. Furthermore, a characterization of the capability of a weapon system in terms of the trajectory information is given.

From a mathematical point of view, the development is trivial, for it merely translates some set theoretic notions and properties into the physical setting.

I. INTRODUCTION

A conventional method to obtain a set of terminal ballistic dispersion data experimentally for the purpose of analyzing the ballistic effectiveness of a weapon system is to fire the weapon upon a sufficiently large target plane. As the result, impact points are secured and upon which a well established analytical procedure is then applied to gain some concrete realization of the effectiveness of the weapon system under evaluation.

The purpose of this study is to establish a reasonable measure of information content in a given trajectory. Hence, such a measure can be applied to the data ensemble. Subsequently, on the basis of this concept and some other necessary mathematical notions, a few simple results are obtained.

The results so obtained give some indications as to what conditions the data ensemble contains all the information offered by a trajectory; in particular, are the data collected from the "plane" giving all the relevent information pertinent to the capability of the weapon system.

From a mathematical point of view, the development is trivial; for it merely translates, for example, some set theoretic notions and properties into a physical setting. Consequently, the theorems obtained are obvious facts.

The reason for reproducing here the last paragraph from the abstract is to remind the readers that the results are some simple natural consequences of mathematics.

II. PRELIMINARIES

In this section we recall some of the well established concepts from the mathematical literatures. Consequently, the proofs of the cited theorems are omitted. Interested readers are referred to any standard mathematical analysis books such as Rudin [4], Dieudonne [1] and Dugundji [2].

Definition: Let R be the set of reals, and A, B be any two non-empty subsets of R^3 (the Enclidean 3-space). The distance between A and B is

$$d(A,B) = \inf \{\rho(a,b) \mid a \in A,\ b \in B\} = \inf_{\substack{a \in A \\ b \in B}} \rho(a,b)$$

where "inf" of C is the greatest lower bound of a non-empty set C and ρ denotes the usual Euclidean distance function on R^3; i.e.,

$$\rho: R^3 \times R^3 \to [0,\infty)$$

defined by

$$\rho(x,y) = \sqrt{\sum_{i=1}^{3} (x_i - y_i)^2}\ , \quad x,y \in R^3$$

where, for example, x_i is the $i^{\underline{th}}$ component of x.

If one of the sets A,B is the null set ϕ, then

$$d(A,B) = \infty .$$

Theorem 2.1: Let A, B be non-empty closed subsets in R^3. Then there exists $a_0 \varepsilon A$ and $b_0 \varepsilon B$ such that

$$d(A,B) = \rho(a_0, b_0)$$

Theorem 2.2: Let f and g be two continuous mappings on a metric space E into a metric space E'. The set A of all points $x \varepsilon E$ such that $f(x) = g(x)$ is closed in E.

Theorem 2.3: If A, B and C are subsets of R^3 such that $B \subset C$, then

$$d(A,B) \geq d(A,C).$$

III. ASSUMPTION AND DEFINITIONS

In this section, we shall state a basic assumption and define some necessary concepts upon which the following development is based.

By the trajectory of a projectile, we take it to be the path generated by the center of mass of the projectile during the flight period. However, in the sequel a more concise definition is needed, we characterize it as follows:

Definition: For each positive integer, let $\Omega_n = \{1,2, \cdots ,n\}$ and $\Gamma = [t_0,t_1]$ be a closed and bounded interval in R, a subset of points $f = (f_1,f_2,f_3)\epsilon R^3$ is called a trajectory F if the following conditions hold:

i) For each $i\epsilon\Omega_3$, f_i is a continuous real-valued function on Γ

ii) There exists an $i\epsilon\Omega_3$ such that f_i is a strictly monotone function on Γ.

The conditions imposed on a trajectory F is quite weak, for in reality the path of a projectile is sufficiently smooth to the degree that f is a differentiable vector-valued function on the interior of Γ; that is, $\dot{f}(t) = (\dot{f}_1(t),\dot{f}_2(t),\dot{f}_3(t))$ exists for every $t\epsilon\Gamma$ and $t_0 \neq t \neq t_1$. The second condition simply illuminates those trajectories containing a "loop" for then there exists $t, t'\epsilon\Gamma$ such that $t \neq t'$ and $f(t) = f(t')$. Also, $\Gamma \neq \phi$ implies that F is not a null trajectory. In view of these observations, we may conclude that the family of all trajectories is large enough to contain the collection of all trajectories generated by non-guided projectiles fired from a launcher (or a weapon system) under experiment.

Definition: For every $i \epsilon \Omega_n$, let ψ_i be a continuous mapping on R^3 into R with respect to the usual distance functions. Then a non-empty subset of points $s = (s_1, s_2, s_3)$ of R^3 is called a surface S if

i) $S = \bigcup_{i=1}^{n} S_i$

ii) $S_i = \{s \epsilon R^3 | \psi_i(s) = k_i \ , \ k_i \epsilon R\}$.

The definition for a surface defined here is different from that which one may find in mathematical literature, for which $n = 1$ and ψ_i is differentiable, [4, p. 209], for example.

Now we have the necessary mathematical tools to introduce some new notions and proceed as follows.

In the course of evaluating experimentally the terminal ballistic effectiveness or capability of a weapon system (particularly small arms) in delivering a projectile to a preassigned fixed "point" (aiming point), we normally collect the terminal dispersion data by placing an aiming point on a sufficiently large plane upon which the weapon system is being fired, Fallin [3]. In this case, the dispersion data are the set of all impact points on the plane, with respect to a fixed two-dimensional cartesian coordinate system.

Every impact point on the plane possesses three intrinsic properties of the corresponding trajectory. Obviously, the three properties are magnitude, direction, and the point, which is itself a point on the trajectory. The vector properties of an impact point provide not only a measure of closeness of a given trajectory to the aiming point but also the bias of the weapon system.

As for the plane, its function is to pick a point on the trajectory, for without such a plane we do not know in general the vector character of a point on a trajectory. In view of this observation, we could use a surface to assume the function of a plane as well. Since every plane is a surface according to the definition above, it would be interesting to find out what kind of surfaces would be best to use instead of a plane. The word "best" calls for some criteria with which a comparison can be made among all the admissible surfaces. The necessity of an appropriate quality measure provides part of the motivations for defining the following concepts.

The foregoing discussions imply that, for each point on a trajectory, there corresponds some measurable quantities indicating the capability of a weapon system in delivering a projectile to the aiming point. A point, for example, on a trajectory possesses some information concerning closeness to the aiming point -- consequently, the accuracy of the system. Obviously, the correspondence between each point on a trajectory and the corresponding information possessed by that point gives in a natural way a function between two sets. Such a function may be defined as follows:

Definition: Let F be a trajectory. A real-valued function I on F into the reals is defined by

$$I(f) = \frac{1}{1 + \rho(f,\theta)} \quad , \quad f \varepsilon F$$

where ρ is the usual distance function on the euclidean three-space, and θ is the aiming point and, for convenience, is taken to be the origin of the coordinate system.

We observe that the function I is well-defined since ρ is. Also, the continuity of I is implied by that of $\rho(\ ,\theta)$ on a subset of R^3. Furthermore, since ρ is a non-negative real-valued function, it follows that $0 \leq I(f) \leq 1$ for any point f belongs to a trajectory F.

Physically, the quantity I(f) is a measure of closeness of a point f on the trajectory F as related to the aiming point. Consequently, it contains knowledge concerning the accuracy of the weapon system under consideration. For these reasons, we may interpret the quantity I(f) as the amount of information associated with F at the point f. The maximum attainable amount of information possessed by a trajectory F is given by

$$\sup_{f\varepsilon F} I(f) = \frac{1}{1 + \inf\limits_{f\varepsilon F} \rho(f,\theta)}$$

Since F is closed and bounded -- the direct image of a continuous mapping on a compact set is compact, we have

$$\sup_{f\varepsilon F} I(f) = \max_{f\varepsilon F} I(f) = \frac{1}{1 + \min\limits_{f\varepsilon F} \ (f,\theta)}$$

Therefore, the maximum amount of information possessed by F is recoverable at some point $f_0\varepsilon F$. Symbolically, $\max\limits_{f\varepsilon F} I(f) = I(f_0)$

(The existence of such a point f_0 is asserted by Theorem 2.1.) As for the uniqueness, we make the following stipulation:

Assumption: There exists at most one point $f_0 \varepsilon F$ such that $I(f_0)$ is the maximum recoverable amount of information of F.

The assumption so stated is not indispensable. However, as a result of this stipulation, subsequent analysis is simplified. In fact, the assumption is reasonable, for in the neighborhood of the aiming point, the radius of curvature of a physically realizable trajectory F at f is greater than $\rho(f_0,\theta)$, if $f \neq f_0$; that is, F is tangent to the sphere

$$\{x \varepsilon R^3 | \rho^2(x,\theta) = \rho^2(f_0,\theta)\}$$

at $f_0 \varepsilon F$.

In view of the foregoing remarks, for each trajectory F there exists one and only one point f_0 belonging to F such that I assumes its maximum value at f_0.

Before defining the last concept for this analysis, we recall for a trajectory F with $g \varepsilon F$, $I(g)$ is the amount of information assigned to F at g. Equivalently, $I(g)$ is the amount of information gained about F by observing (sampling) g. Having this interpretation in mind, we make the following definition.

Definition: Let F be a trajectory. A real valued function $\overline{I}$ on F into the unit interval is defined by the formula

$$\overline{I}(g) = \max_{f \varepsilon F} I(f) - I(g) \; , \; g \varepsilon F$$

We note that $0 \leq \overline{I}(g) \leq I(f_0)$ and $\overline{I}(g) = 0$ if and only if $g = f_0$. It is clear that $\overline{I}(g)$ is the amount of information lost concerning F due to observing g.

Remark: If, in the definition of a trajectory, t_0 is taken to be the time at which the projectile leaves the muzzle point of a weapon, then the muzzle velocity $\dot{f}(t_0)$ is a well defined quantity. Hence, f is differentiable at t_0 also. Furthermore, the definition does not account for the case in which a round is misfired due to, for instance, malfunction of the weapon. In such an event, the trajectory is the null set in R^3.

IV. MAIN RESULTS

With the foregoing concepts at our disposal, we may deduce a few trivial consequences. The proof of the first theorem is immediate and is omitted.

Theorem 4.1: Let $\{F_i | i\varepsilon\Omega_n\}$ be a set of trajectories and, for each $i\varepsilon\Omega_n$, f_{i0} be the unique point of F_i for which $\max_{f\varepsilon F_i} I(f) = I(f_{i0})$. Then

$$\sum_{i=1}^{n} I(f_{i0}) = n$$

if and only if everyone of the n trajectories hits the aiming point θ.

Before we proceed to the second theorem, we need the following fact at our disposal.

Lemma: Every surface is a closed set.

Proof: By definition a surface S is a non-empty subset of R^3 such that

$$S = \bigcup_{i=1}^{n} S_i, \quad \text{for some positive integer } n, \text{ and for } i\varepsilon\Omega_n$$

$$S_i = \{s\varepsilon R^3 | \psi_i(s) = k_i; \ \ k_i\varepsilon R\}.$$

Now by definition ψ_i is a continuous function on R^3 into R, and the constant function C_i defined by

$$C_i(s) = k_i \ , \ s\varepsilon R^3$$

is also a continuous mapping on R^3 into R. By Theorem 2.2, the set

$$\{s\varepsilon R^3 | \psi_i(s) = C_i(s)\}$$

is closed and is precisely S_i, for $C_i(s) = k_i$. The assertion follows from the fact that a finite union of closed set is closed.

<u>Theorem 4.2</u>: Let $\{F_i | i\varepsilon\Omega_n\}$ be the set of trajectories corresponding to the n rounds fired from a weapon system at an aiming point, and S be a surface. Furthermore, let A_i be defined by

$$A_i = \{g\varepsilon R^3 | g\varepsilon S \cap F_i\}.$$

Then

$$\sum_{i=1}^{n} \max_{g\varepsilon A_i} I(g) = \sum_{i=1}^{n} \max_{g\varepsilon F_i} I(g)$$

is a necessary and sufficient condition for no information is being lost by the experiment.

<u>Proof</u>: Sufficiency

$A_i \subset F_i$ implies $\inf\limits_{g\varepsilon F_i} \rho(g,\theta) \leq \inf\limits_{g\varepsilon A_i} \rho(g,\theta)$ by Theorem 2.3.

Since $\rho(g,\theta) \geq 0$. We have

$$\frac{1}{1 + \inf\limits_{g\varepsilon A_i} \rho(g,\theta)} \leq \frac{1}{1 + \inf\limits_{g\varepsilon F_i} \rho(g,\theta)},$$

or equivalently

$$\sup_{g\epsilon A_i} I(g) \leq \sup_{g\epsilon F_i} I(g).$$

Therefore, for each $i\epsilon\Omega_n$

$$\sup_{g\epsilon F_i} I(g) - \sup_{g\epsilon A_i} I(g) \geq 0.$$

By the lemma above, S is closed and so is F_i. It follows that $A_i = S\cap F_i$ is closed. Hence, supremum is the same as maximum. We have

$$\max_{g\epsilon F_i} I(g) - \max_{g\epsilon A_i} I(g) \geq 0.$$

Now, the hypothesis implies that for each $i\epsilon\Omega_n$

$$\max_{g\epsilon F_i} I(g) - \max_{g\epsilon A_i} I(g) = 0.$$

If $A_i \neq \phi$ for each $i\epsilon\Omega_n$, then there exists $g_{i0}\epsilon A_i$ such that

$$I(g_{i0}) = \max_{g\epsilon A_i} I(g)$$

Therefore

$$\overline{I}(g_{i0}) = \max_{g\epsilon F_i} I(g) - I(g_{i0}) = 0.$$

Thus, the amount of information lost concerning F_i due to observing g_{i0} is zero, and the sum

$$\sum_{i=1}^{n} \bar{I}(g_{i0}) = 0$$

vanishes asserts that no information is being lost by the experiment.

Now suppose for some $i\epsilon\Omega_n$, $A_i = \Phi$. Then

$$\max_{g\epsilon A_i} I(g) = \sup_{g\epsilon A_i} I(g) = 0$$

The first equality follows from the fact that the null set Φ is closed and the convention of the supremum over the null set yields the last equality. So for this i,

$$\max_{g\epsilon F_i} I(g) - \max_{g\epsilon A_i} I(g) = \max_{g\epsilon F_i} I(g) > 0$$

The strict inequality follows from a remark in section 3 that $F_i \neq \Phi$. Thus if $A_i = \Phi$ for some $i\epsilon\Omega_n$, then the hypothesis does not hold. We may then deduce that the conclusion is true.

It remains to prove the necessary condition. No information is lost and the fact that $F_i \neq \Phi$, $i\epsilon\Omega_n$, imply $A_i \neq \Phi$. Then there exists $f_{i0}\epsilon A_i$ such that

$$\max_{g\epsilon A_i} I(g) = I(f_{i0}).$$

Also

$$0 = \max_{g\epsilon F_i} I(g) - I(f_{i0}) = \max_{g\epsilon F_i} I(g) - \max_{g\epsilon A_i} I(g)$$

Summing over the last term on i gives the assertion, and the proof of the theorem is complete.

As the immediate consequences of this result, we have

Corollary 4.3: Suppose A_i for each $i \epsilon \Omega_n$ contains one and only one point. There exists at least one $i \epsilon \Omega_n$ such that, for $g_i \epsilon A_i$, $\bar{I}(g_i) > 0$ if and only if

$$\sum_{i=1}^{n} \max_{f \epsilon F_i} I(f) > \sum_{i=1}^{n} I(g_i).$$

Corollary 4.4: Let S and S^* be two surfaces such that $F_i \cap S$ is a proper subset of $F_i \cap S^*$ for at least one i. Then, for a set of n trajectories,

$$\sum_{i=1}^{n} \max_{f \epsilon F_i \cap S} I(f) \leq \sum_{i=1}^{n} \max_{f \epsilon F_i \cap S^*} I(f) \leq \sum_{i=1}^{n} \max_{f \epsilon F_i} I(f)$$

Theorem 4.5: Let F be a trajectory and S be a surface and A be the intersection of F and S. Suppose

$$\max_{f \epsilon A} I(f) < \max_{f \epsilon F} I(f)$$

Then there exists a sequence $\{h_i\}_{i=1}^{\infty} \subset F$ such that

i) $\max_{f \epsilon F} I(f) = \lim_{i \to \infty} I(h_i)$

ii) there is a positive integer N such that, for all

$i \geq N$, $I(h_i) > \max_{f \epsilon A} I(f)$

Proof: For $F \neq \Phi$ Theorem 2.1 asserts that there is a point $f_0 \varepsilon F$ such that

$$\max_{f \varepsilon F} I(f) = I(f_0).$$

Since F is closed, then there exists a sequence $\{h_i\}_{i=1}^{\infty} \subset F$ such that $\lim_{i\to\infty} h_i = f_0$. Also we have seen in section 3 that the function I is continuous; it is this property for which the following first equality holds.

$$\lim_{i\to\infty} I(h_i) = I(\lim_{i\to\infty} h_i) = I(f_0) = \max_{f\varepsilon F} I(f).$$

It remains to be shown the last assertion of the theorem. If $A = \Phi$, ii) holds trivially, since $F \neq \Phi$ and we may pick $N = 1$. Suppose $A \neq \Phi$. Again, Theorem 2.1 implies there is a point $g_0 \varepsilon A$ such that

$$\max_{f \varepsilon A} I(f) = I(g_0).$$

Now, by hypothesis we obtain $\rho(f_0,\theta) < \rho(g_0,\theta)$. Also we may pick an $\varepsilon > 0$ so that $\rho(f_0,\theta) + \varepsilon < \rho(g_0,\theta)$. For this ε and the fact that

$$\lim_{i\to\infty} h_i = f_0$$

We conclude that there exists a positive integer N such that for all $i \geq N$

$$\rho(h_i,f_0) < \varepsilon.$$

Using these facts and the well-known properties of the metric ρ, we obtain

$$\rho(h_i,\theta) < \rho(h_i,f_0) + \rho(f_0,\theta) < \varepsilon + \rho(f_0,\theta) < \rho(g_0,\theta).$$

Obviously, $1 + \rho(h_i,\theta)$, $1 + \rho(g_0,\theta) > 0$. It follows that

$$\frac{1}{1 + \rho(h_i,\theta)} > \frac{1}{1 + \rho(g_0,\theta)} = I(g_0) = \max_{f \epsilon A} I(f)$$

for all $i \geq N$, where N depends on ϵ.

Remark: We recall that in section 3 the definition of a surface requires every surface being non-empty. The restriction does not impose any further constraint on the results in this section; that is, the theorems and their consequences remain valid for the case in which S is empty and in particular the lemma remains true since the null set is closed. A trivial example of a null surface is given by

$$S = \{x \epsilon R^3 | \rho^2(ax,bx_0) = -1, \ a, \ b \epsilon R, \ x_0 \epsilon R^3\}.$$

V. CONCLUSIONS

We conclude the study by stating explicitly some of the observations deduced from the foregoing results.

For a given experiment,

$$\sum_{i=1}^{n} \max_{f \epsilon F_i} I(f)$$

is the total of the maximum recoverable amount of information possessed by a set of n trajectories. Theorem 4.1 states that such a total is equal to n if and only if the weapon system gives a perfect performance concerning its capability in delivering the n projectiles to the aiming point.

Theorem 4.2 asserts that, in the course of evaluating the terminal ballistic effectiveness of a weapon based on the results of analyzing the terminal dispersion data collected from the surface, an "ideal surface" to be used is one which perserves all the information offered by the experiment; equivalently, an ideal surface is one having a set of n trajectories as its subset. A simple example of such a surface S_I is

$$S_I = \bigcup_{\lambda \epsilon [\alpha,\beta]} \{(x_1,x_2,x_3) \epsilon R^3 | x_1 = \lambda;\ x_2,\ x_3 \epsilon R\}$$

where

$$\alpha = \min \Lambda_0$$

$$\beta = \max \Lambda_1$$

$$\Lambda_j = \{f_i(t_j) | f_i \epsilon F_i,\ i \epsilon \Omega_n\}\ ,\ j=0,1$$

According to our definition of a surface, S_I is not a surface, for S_I is a union of uncountably many planes. Therefore, a perfect surface does not exist. Also, the second consequence of the theorems tells us that there is a positive information gained by extending the surface being used in such a way there is an increase in the data ensemble for at least one trajectory.

Finally, the last result concludes that there is a sequence of points in a given trajectory such that the limit of the image of the sequence under I is precisely the maximum information possessed by the trajectory. Also, if there is a loss of information, such a loss can be reduced by using an algorithm to recover some of the loss.

Remark: In view of S_I defined above, a reasonable surface to be used in an experiment is one which consists of $2m+1$ sufficiently large planes having the aiming point placed on the center of one plane and the remaining $2m$ planes erected at equidistance from each other on both sides of the center plane.

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With the use of this concept and some appropriate mathematical tools a few immediate results are obtained. Furthermore, a characterization of the capability of a weapon system in terms of the trajectory information is given.

From mathemtical point of view, the development is trivial, for it merely translates some set theoretic notions and properties into the physical setting.

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Information						
Trajectory						
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Accuracy						
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